Using Jones's Trick to Solve for Steady State Growth Rates

Balanced growth requires that endogenous aggregate variables like Y and K grow at common rates in the steady state, so that the ratio K/Y should be constant. If we accept this at the outset as a necessary feature of the steady state it becomes possible to quickly deduce steady state growth rates from the production function alone. Jones uses this trick in Chapter 9 to facilitate the derivation of growth rates in the Solow model with natural resources as a third factor of production. This handout begins by verifying that the trick works in some of the versions of the Solow model with which we are already familiar: without technical change, with technical change, and under the MRW specification of human capital. It then proceeds to models relevant to the analysis of Chapter 9.

1. Without Technical Change

The production function is $Y = K^\alpha L^{1-\alpha}$.

Transform this to a version in which the ratio K/Y appears:

$$\frac{Y}{Y^\alpha} = \frac{K^\alpha}{Y^\alpha} L^{1-\alpha}$$

or

$$Y^{1-\alpha} = \left(\frac{K}{Y}\right)^\alpha L^{1-\alpha}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha)$:

$$Y = \left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)} L.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{\alpha}{1-\alpha} \log\left(\frac{K}{Y}\right) + \log L$$

$$\frac{\mathrm{d}\log Y}{\mathrm{d}t} = \frac{\alpha}{1-\alpha} \frac{\mathrm{d}\log(K/Y)}{\mathrm{d}t} + \frac{\mathrm{d}\log L}{\mathrm{d}t}$$

Using the premise that the ratio K/Y should be constant in a steady state we conclude that

$$\frac{\dot{Y}}{Y} = \frac{\alpha}{1-\alpha} \cdot 0 + \frac{\dot{L}}{L} = 0 + n = n,$$

which we know to be the correct steady state growth rate for Y in the Solow model without technical change. From this we can obtain the growth rates of any other variables of interest, such as output per worker $y = Y/L$:

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = n - n = 0.$$

2. With Labour-Augmenting Technical Change

The production function is $Y = K^\alpha (AL)^{1-\alpha}$.

Transform this to a version in which the ratio K/Y appears:

$$Y^{1-\alpha} = \left(\frac{K}{Y}\right)^\alpha (AL)^{1-\alpha}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha)$:

$$Y = \left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)} AL.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{\alpha}{1-\alpha} \log\left(\frac{K}{Y}\right) + \log A + \log L$$
$$\frac{\mathrm{d} \log Y}{\mathrm{d}t} = \frac{\alpha}{1-\alpha} \frac{\mathrm{d} \log(K/Y)}{\mathrm{d}t} + \frac{\mathrm{d} \log A}{\mathrm{d}t} + \frac{\mathrm{d} \log L}{\mathrm{d}t}$$

Using the premise that the ratio K/Y should be constant in a steady state we conclude that

$$\frac{\dot{Y}}{Y} = \frac{\alpha}{1-\alpha} \cdot 0 + \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = 0 + g_A + n = g_A + n,$$

which we know to be the correct steady state growth rate for Y in the Solow model with technical change. From this we can obtain the growth rates of any other variables of interest, such as output per worker $y = Y/L$:

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = g_A + n - n = g_A.$$

This is the classic result that technical change is the engine of growth: it is the fundamental source of sustained growth in living standards.

3. Alternative Specification for Technical Change

The above result is not sensitive to the specification of technical change. Consider the alternative specification of the production function as

$$Y = BK^{\alpha}L^{1-\alpha} \qquad \text{where } B \equiv A^{1-\alpha}.$$

The growth rates of B and A are related by

$$g_B \equiv \frac{\dot{B}}{B} = (1-\alpha)\frac{\dot{A}}{A} = (1-\alpha)g_A.$$

Transform the production function to a version in which the ratio K/Y appears:

$$Y^{1-\alpha} = B\left(\frac{K}{Y}\right)^{\alpha} L^{1-\alpha}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha)$:

$$Y = B^{1/(1-\alpha)}\left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)} AL.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{1}{1-\alpha}\log B + \frac{\alpha}{1-\alpha}\log\left(\frac{K}{Y}\right) + \log L$$

$$\frac{\mathrm{d}\log Y}{\mathrm{d}t} = \frac{1}{1-\alpha}\frac{\mathrm{d}\log B}{\mathrm{d}t} + \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log(K/Y)}{\mathrm{d}t} + \frac{\mathrm{d}\log L}{\mathrm{d}t}$$

Using the premise that the ratio K/Y should be constant in a steady state we conclude that

$$\frac{\dot{Y}}{Y} = \frac{1}{1-\alpha}\frac{\dot{B}}{B} + \frac{\alpha}{1-\alpha}\cdot 0 + \frac{\dot{L}}{L} = \frac{1}{1-\alpha}g_B + 0 + n = \frac{1}{1-\alpha}g_B + n,$$

Of course this can be restated in terms of g_A if desired, yielding the usual result

$$\frac{\dot{Y}}{Y} = \frac{1}{1-\alpha}(1-\alpha)g_A + n = g_A + n.$$

4. The Mankiw-Romer-Weil Approach to Human Capital

The production function is $Y = K^\alpha H^\beta (AL)^{1-\alpha-\beta}$.

Both K and H are accumulable factors and play symmetric roles in the model. Thus, by the same logic that K/Y should be constant in a steady state, so should H/Y.

We therefore proceed by transforming the production function to a version in which the ratios K/Y and H/Y appear:

$$\frac{Y}{Y^{\alpha+\beta}} = \frac{K^\alpha}{Y^\alpha}\frac{H^\beta}{Y^\beta}(AL)^{1-\alpha-\beta}$$

or

$$Y^{1-\alpha-\beta} = \left(\frac{K}{Y}\right)^\alpha \left(\frac{H}{Y}\right)^\beta (AL)^{1-\alpha-\beta}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha-\beta)$:

$$Y = \left(\frac{K}{Y}\right)^{\alpha/(1-\alpha-\beta)} \left(\frac{H}{Y}\right)^{\beta/(1-\alpha-\beta)} AL.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{\alpha}{1-\alpha-\beta}\log\left(\frac{K}{Y}\right) + \frac{\beta}{1-\alpha-\beta}\log\left(\frac{H}{Y}\right) + \log A + \log L$$

$$\frac{\mathrm{d}\log Y}{\mathrm{d}t} = \frac{\alpha}{1-\alpha-\beta}\frac{\mathrm{d}\log(K/Y)}{\mathrm{d}t} + \frac{\beta}{1-\alpha-\beta}\frac{\mathrm{d}\log(H/Y)}{\mathrm{d}t} + \frac{\mathrm{d}\log A}{\mathrm{d}t} + \frac{\mathrm{d}\log L}{\mathrm{d}t}$$

Using the premise that the ratios K/Y and H/Y should be constant in a steady state we conclude that

$$\frac{\dot{Y}}{Y} = \frac{\alpha}{1-\alpha-\beta}\cdot 0 + \frac{\beta}{1-\alpha-\beta}\cdot 0 + \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = 0 + 0 + g_A + n = g_A + n,$$

which we know to be the correct steady state growth rate for Y in this model. Thus we find that the conclusion that living standards grow at the rate of labour-augmenting technical change, $\dot{y}/y = g_A$, is robust to the introduction of human capital.

5. Natural Resources that Do Not Become Increasingly Scarce

Formulated in terms of B

Consider a natural resource H that does not become increasingly scarce in the sense that it remains in constant proportion ϕ to L: $H/L = \phi$. It follows that H grows at the same rate as L: $\dot{H}/H = n$.

The production function is

$$Y = BK^{\alpha}H^{\beta}L^{1-\alpha-\beta}$$

where $B = A^{1-\alpha-\beta}$ and so

$$g_B \equiv \frac{\dot{B}}{B} = (1-\alpha-\beta)\frac{\dot{A}}{A} = (1-\alpha-\beta)g_A.$$

(Note: Although H is not interpreted as human capital, formally this model combines the MRW three-factor production function with Jones's specification of human capital, $H = e^{\psi u}L$. We have introduced the notational simplification $\phi = e^{\psi u}$.)

Now K and H no longer play symmetric roles in the model and so, although K/Y is still assumed to be constant in a steady state, there is no reason to assume this to be true of H/Y a priori.

We therefore proceed by transforming the production function to a version in which the ratio K/Y appears:

$$\frac{Y}{Y^{\alpha}} = B\frac{K^{\alpha}}{Y^{\alpha}}H^{\beta}L^{1-\alpha-\beta}$$

or

$$Y^{1-\alpha} = B\left(\frac{K}{Y}\right)^{\alpha}H^{\beta}L^{1-\alpha-\beta}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha)$:

$$Y = B^{1/(1-\alpha)}\left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)}H^{\beta/(1-\alpha)}L^{(1-\alpha-\beta)/(1-\alpha)}.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{1}{1-\alpha}\log B + \frac{\alpha}{1-\alpha}\log\left(\frac{K}{Y}\right) + \frac{\beta}{1-\alpha}\log H + \frac{1-\alpha-\beta}{1-\alpha}\log L$$
$$\frac{\mathrm{d}\log Y}{\mathrm{d}t} = \frac{1}{1-\alpha}\frac{\mathrm{d}\log B}{\mathrm{d}t} + \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log(K/Y)}{\mathrm{d}t} + \frac{\beta}{1-\alpha}\frac{\mathrm{d}\log H}{\mathrm{d}t} + \frac{1-\alpha-\beta}{1-\alpha}\frac{\mathrm{d}\log L}{\mathrm{d}t}$$

Using the premise that the ratio K/Y should be constant in a steady state, and defining the notation $\bar{\beta} = \beta/(1-\alpha)$, we conclude that

$$\begin{aligned}\frac{\dot{Y}}{Y} &= \frac{1}{1-\alpha}\frac{\dot{B}}{B} + \frac{\alpha}{1-\alpha}\cdot 0 + \bar{\beta}\frac{\dot{H}}{H} + (1-\bar{\beta})\frac{\dot{L}}{L} \\ &= \frac{1}{1-\alpha}g_B + \bar{\beta}n + (1-\bar{\beta})n \\ &= \frac{1}{1-\alpha}g_B + n.\end{aligned}$$

Alternatively, restated in terms of g_A this is

$$\frac{\dot{Y}}{Y} = \frac{1-\alpha-\beta}{1-\alpha}g_A + n = (1-\bar{\beta})g_A + n,$$

which reduces to the familiar growth rates of the textbook model in the appropriate special cases:

$$\frac{\dot{Y}}{Y} = (1-\bar{\beta})g_A + n = \begin{cases} g_A + n & \text{if } \beta = 0 \\ n & \text{if } g_A = 0. \end{cases}$$

The steady state growth rate of output per worker is

$$\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - n = (1 - \bar{\beta})g_A \qquad (\equiv g \text{ in Jones's notation.})$$

Note that the "g" of Jones's notation may be defined alternatively as

$$g \equiv (1 - \bar{\beta})g_A = \frac{1}{1-\alpha} g_B.$$

Its interpretation is as the growth rate of output per worker in the presence of natural resources that are not becoming increasingly scarce.

Note further that $\bar{\beta}$ may be written

$$\bar{\beta} = \frac{\beta}{1-\alpha} = \frac{\beta}{\beta + (1 - \alpha - \beta)}.$$

Recall that β and $1 - \alpha - \beta$ are each factor shares (of H and L, respectively) and so satisfy $0 < \beta < 1$ and $0 < 1 - \alpha - \beta < 1$. It follows that $\bar{\beta}$ is a positive fraction: $0 < \bar{\beta} < 1$.

Thus there is a **growth drag** of natural resources,

$$\frac{\dot{y}}{y} = (1 - \bar{\beta})g_A < g_A, \qquad \text{because } 0 < 1 - \bar{\beta} < 1,$$

even when the natural resource does not become increasingly scarce. In the special case in which the natural resource is unimportant in production and so has a factor share of zero ($\beta = 0$ and so $\bar{\beta} = 0$) the steady state growth rate reduces to the familiar

$$\frac{\dot{y}}{y} = (1 - \bar{\beta})g_A = g_A, \qquad \text{if } \bar{\beta} = \beta = 0.$$

What kind of variable is constant in the steady state of this model? It is

$$\frac{y}{A^{1-\bar{\beta}}} = \frac{Y}{A^{1-\bar{\beta}}L}.$$

To see this take logs and time derivatives to obtain

$$\frac{\dot{y}}{y} - (1 - \bar{\beta})\frac{\dot{A}}{A} = (1 - \bar{\beta})g_A - (1 - \bar{\beta})g_A = 0.$$

This indicates why it would be difficult to solve the model by transforming it to a form that is in terms of such variables, and hence the value of Jones's modeling trick.

6. Renewable Natural Resources

Consider a natural resource T that is constant.

The production function is

$$Y = BK^\alpha T^\beta L^{1-\alpha-\beta}$$

where $B = A^{1-\alpha-\beta}$ and so

$$g_B \equiv \frac{\dot{B}}{B} = (1-\alpha-\beta)\frac{\dot{A}}{A} = (1-\alpha-\beta)g_A.$$

We proceed by transforming the production function to a version in which the ratio K/Y appears:

$$Y^{1-\alpha} = B\left(\frac{K}{Y}\right)^\alpha T^\beta L^{1-\alpha-\beta}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha)$:

$$Y = B^{1/(1-\alpha)}\left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)} T^{\beta/(1-\alpha)} L^{(1-\alpha-\beta)/(1-\alpha)}.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{1}{1-\alpha}\log B + \frac{\alpha}{1-\alpha}\log\left(\frac{K}{Y}\right) + \frac{\beta}{1-\alpha}\log T + \frac{1-\alpha-\beta}{1-\alpha}\log L$$

$$\frac{\mathrm{d}\log Y}{\mathrm{d}t} = \frac{1}{1-\alpha}\frac{\mathrm{d}\log B}{\mathrm{d}t} + \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log(K/Y)}{\mathrm{d}t} + \frac{\beta}{1-\alpha}\frac{\mathrm{d}\log T}{\mathrm{d}t} + \frac{1-\alpha-\beta}{1-\alpha}\frac{\mathrm{d}\log L}{\mathrm{d}t}$$

Using the premise that the ratio K/Y should be constant in a steady state, and defining the notation $\bar{\beta} = \beta/(1-\alpha)$, we conclude that

$$\begin{aligned}
\frac{\dot{Y}}{Y} &= \frac{1}{1-\alpha}\frac{\dot{B}}{B} + \frac{\alpha}{1-\alpha}\cdot 0 + \bar{\beta}\cdot 0 + (1-\bar{\beta})\frac{\dot{L}}{L} \\
&= \frac{1}{1-\alpha}g_B + (1-\bar{\beta})n \\
&= \frac{1-\alpha-\beta}{1-\alpha}g_A + (1-\bar{\beta})n \\
&= (1-\bar{\beta})(g_A+n).
\end{aligned}$$

In the special case in which the natural resource is unimportant in production and so has a factor share of zero ($\beta = 0$ and so $\bar{\beta} = 0$) this reduces to the familiar

$$\frac{\dot{Y}}{Y} = (1-\bar{\beta})(g_A+n) = g_A + n \qquad \text{if } \beta = 0.$$

In general, the steady state growth rate of output per worker is

$$\begin{aligned}
\frac{\dot{y}}{y} &= \frac{\dot{Y}}{Y} - n = (1-\bar{\beta})(g_A+n) - n \\
&= \begin{cases} g_A - \bar{\beta}(g_A+n) < g_A & (1) \\ (1-\bar{\beta})g_A - \bar{\beta}n = g - \bar{\beta}n < g & (2) \quad \text{where ``}g\text{''} = (1-\bar{\beta})g_A \end{cases}
\end{aligned}$$

These expressions describe the growth drag of resource scarcity; expression (2) is the version found in Jones (p. 172, last displayed equation). The size of the growth drag depends on the benchmark against which it is compared:

Expression (1) compares $\dot{y}/y$ against the growth rate g_A of the textbook Solow model.
Expression (2) compares $\dot{y}/y$ against the growth rate "g"$= (1-\bar{\beta})g_A$ of the model with a resource that is not subject to increasing scarcity.

In either case the growth drag is negative. In expression (1) it is particularly easy to see that the magnitude of this negative drag becomes larger with the factor share β, which represents the importance of the scarce resource in production.

7. Nonrenewable Natural Resources

Consider a natural resource that is nonrenewable. It has stock R from which flow E is extracted per unit time. In continuous time R is therefore depleted as $\dot{R} = -E$. The simplest specification for the rate of extraction is to assume that the flow E is a constant proportion of the stock R denoted s_E:

$$s_E = \frac{E}{R}.$$

Thus if $s_E = 0.01$ it means that 1% of the resource stock is extracted each period. Using $\dot{R} = -E$ it follows that

$$\frac{\dot{R}}{R} = -s_E,$$

so that the stock declines at instantaneous growth rate -0.01.

Finally, we have that $E = s_E R$ and so, given that the extraction rate s_E is assumed constant,

$$\frac{\dot{E}}{E} = \frac{\dot{R}}{R} = -s_E.$$

Thus both the stock $R(t)$ and the flow extracted from it, $E(t)$, decline continuously at the instantaneous negative growth rate $-s_E$. They therefore evolve in their levels according to continuous growth processes similar to what we have used for labour and technology, the only difference being that now the growth rate is negative. In the case of the stock $R(t)$, for example, we can write

$$R(t) = R(0)e^{-s_E t}.$$

It is the flow E that enters the production function:

$$Y = BK^\alpha E^\gamma L^{1-\alpha-\gamma}$$

where $B = A^{1-\alpha-\gamma}$ and so, similar to before,

$$g_B \equiv \frac{\dot{B}}{B} = (1-\alpha-\gamma)\frac{\dot{A}}{A} = (1-\alpha-\gamma)g_A.$$

We proceed much as before, transforming the production function to a version in which the ratio K/Y appears:

$$Y^{1-\alpha} = B\left(\frac{K}{Y}\right)^\alpha E^\gamma L^{1-\alpha-\gamma}.$$

Isolate Y on the left hand side by raising both sides to the power $1/(1-\alpha)$:

$$Y = B^{1/(1-\alpha)}\left(\frac{K}{Y}\right)^{\alpha/(1-\alpha)} E^{\gamma/(1-\alpha)} L^{(1-\alpha-\gamma)/(1-\alpha)}.$$

Now take logarithms and time-derivatives.

$$\log Y = \frac{1}{1-\alpha}\log B + \frac{\alpha}{1-\alpha}\log\left(\frac{K}{Y}\right) + \frac{\gamma}{1-\alpha}\log E + \frac{1-\alpha-\gamma}{1-\alpha}\log L$$

$$\frac{\mathrm{d}\log Y}{\mathrm{d}t} = \frac{1}{1-\alpha}\frac{\mathrm{d}\log B}{\mathrm{d}t} + \frac{\alpha}{1-\alpha}\frac{\mathrm{d}\log(K/Y)}{\mathrm{d}t} + \frac{\gamma}{1-\alpha}\frac{\mathrm{d}\log E}{\mathrm{d}t} + \frac{1-\alpha-\gamma}{1-\alpha}\frac{\mathrm{d}\log L}{\mathrm{d}t}$$

Using the premise that the ratio K/Y should be constant in a steady state, and defining the notation $\bar{\gamma} = \gamma/(1-\alpha)$, we conclude that

$$\begin{aligned}
\frac{\dot{Y}}{Y} &= \frac{1}{1-\alpha}\frac{\dot{B}}{B} + \frac{\alpha}{1-\alpha}\cdot 0 - \bar{\gamma}s_E + (1-\bar{\gamma})\frac{\dot{L}}{L} \\
&= \frac{1}{1-\alpha}g_B - \bar{\gamma}s_E + (1-\bar{\gamma})n \\
&= \frac{1-\alpha-\gamma}{1-\alpha}g_A - \bar{\gamma}s_E + (1-\bar{\gamma})n \\
&= (1-\bar{\gamma})(g_A+n) - \bar{\gamma}s_E.
\end{aligned}$$

In the special case in which the natural resource is unimportant in production and so has a factor share of zero ($\gamma = 0$ and so $\bar{\gamma} = 0$) this reduces to the familiar

$$\frac{\dot{Y}}{Y} = (1-\bar{\gamma})(g_A + n) - \bar{\gamma} s_E = g_A + n \qquad \text{if } \gamma = 0.$$

Similarly, in this special case the growth rate output per worker is $\dot{y}/y = g_A$, which we know to be correct for the textbook model.

In general the steady state growth rate of output per worker is

$$\begin{aligned}
\frac{\dot{y}}{y} &= \frac{\dot{Y}}{Y} - n \\
&= (1-\bar{\gamma})(g_A + n) - \bar{\gamma} s_E - n \\
&= (1-\bar{\gamma}) g_A - \bar{\gamma}(n + s_E) \\
&< \begin{cases} g_A, & \text{the growth rate in the absence of natural resources } (\gamma = 0) \\ (1-\bar{\gamma}) g_A & \equiv g, \text{ the growth rate when the natural resource does not become increasingly scarce} \\ (1-\bar{\gamma}) g_A - \bar{\gamma} n & \text{the growth rate when the natural resource is renewable} \end{cases}
\end{aligned}$$

These inequalities follow because, just as we reasoned previously for $\bar{\beta}$, $\bar{\gamma}$ is a positive fraction: $0 < \bar{\gamma} < 1$. Thus the growth drag of a nonrenewable resource is even greater than the growth drags arising from the other types of natural resources we have studied.

This general expression for $\dot{y}/y$ yields a new comparative statics result: an increase in the extraction rate s_E reduces steady state $\dot{y}/y$.